math

we consider m asset on n periods.

portfolio: $\mathbf{b} = [b_1, \ldots, b_m]^\top$, where b_i denotes the proportion of wealth invested on asset i.

relative price vector on period i: $\mathbf{x}_i = [x_{i,1}, \ldots, x_{i,m}]^\top$, where $x_{i,j}$ denotes the relative price of asset j on period i.

price matrix $X = [\mathbf{x_1}, \ldots \mathbf{x_n}] \in \mathbb{R}^{m \times n}$.

mean of returns on periods:

$$\mu_b = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x_i}^\top \mathbf{b} = \bar{\mathbf{x}}^\top \mathbf{b}, \\ where\ \bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x_i}.$$

variance of returns on periods:

$$\begin{aligned} \sigma_b^2 &= E[(\mathbf{x}_i^\top \mathbf{b} - \bar{\mathbf{x}}^\top \mathbf{b})^2] \\ &= \frac{1}{n-1}\sum_{i=1}^{n}[(\mathbf{x}_i^\top - \bar{\mathbf{x}}^\top)\mathbf{b}]^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{b}^\top(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i^\top - \bar{\mathbf{x}}^\top)\mathbf{b} \\ &= \mathbf{b}^\top \Sigma \mathbf{b} \\ where\ \Sigma &= \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i^\top - \bar{\mathbf{x}}^\top) = conv(\mathbf{X}). \end{aligned}$$

we formulate it as an optimization problem:

$$\begin{aligned} \mathit{minimize}\ & \lambda \mathbf{b}^\top \Sigma \mathbf{b} - \bar{\mathbf{x}}^\top \mathbf{b} \\ s.t.\ & \mathbf{b}^\top \mathbf{1} = 1, \\ & \mathbf{b} \succeq \mathbf{0} \end{aligned}$$

where λ is the risk-aversion parameter.