math

we consider m asset on n periods.

portfolio: $\mathbf{b} = [b_1, \ldots, b_m]^\top$, where b_i denotes the proportion of wealth invested on asset i.

relative price vector on period i: $\mathbf{x}_i = [x_{i,1}, \ldots, x_{i,m}]^\top$, where $x_{i,j}$ denotes the relative price of asset j on period i.

price matrix $X = [\mathbf{x}_1, \ldots \mathbf{x_n}] \in \mathbb{R}^{m \times n}$.

mean of returns on periods:

$$\mu_b = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x_i}^\top \mathbf{b} = \bar{\mathbf{x}}^\top \mathbf{b},$$
$$where\ \bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x_i}.$$

variance of returns on periods:

$$
\begin{aligned}
\sigma_b^2 &= E((\mathbf{x}_i^\top \mathbf{b})^2) - [E(\mathbf{x}_i^\top \mathbf{b})]^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i^\top \mathbf{b})^2 - (\bar{\mathbf{x}}^\top \mathbf{b})^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathbf{b}^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{b} - \mathbf{b}^\top \bar{\mathbf{x}}\bar{\mathbf{x}}_i^\top \mathbf{b} \\
&= \mathbf{b}^\top (\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \mathbf{x}_i^\top - \bar{\mathbf{x}}\bar{\mathbf{x}}_i^\top)\mathbf{b} \\
&= \mathbf{b}^\top \Sigma \mathbf{b}, \\
\textit{where } \Sigma &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \mathbf{x}_i^\top - \bar{\mathbf{x}}\bar{\mathbf{x}}_i^\top.
\end{aligned}
$$

we formulate it as an optimization problem:

$$
\begin{aligned}
\textit{minimize } & \lambda \mathbf{b}^\top \Sigma \mathbf{b} - \bar{\mathbf{x}}^\top \mathbf{b} \\
s.t.\ & \mathbf{b}^\top \mathbf{1} = 1, \\
& \mathbf{b} \succeq \mathbf{0}
\end{aligned}
$$

where λ is the risk-aversion parameter.